‘The Tensor Calculus Knows Physics Better Than the Physicist’

Bachelard on the Role of ‘Covariant Differentiation’ in Relativity Theory

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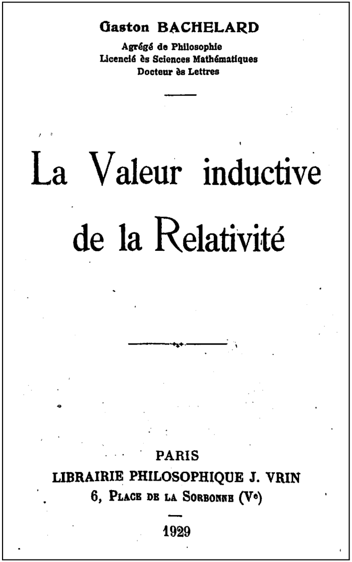
Introduction



Gaston BACHELARD

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La Valeur inductive
de la Relativité

PARIS
LIBRAIRIE PHILOSOPHIQUE J. VRIN
6, Place de la Sorbonne (Ve)

1929

- first published in 1929, the **least cited** among his works
  - obscured by Meyerson and emergence of quantum mechanics
  - Bachelard moved on and disowned it

- **renewed attention** after a 2014 reprinting
  - Bachelard scholarship (Fruteau De Laclos, Parrochia, Alunni)
  - relativity scholarship (Hentschel, Friedman, Ryckman, etc.)

Introduction

- Reichenbach (1928) $\Longrightarrow$ general relativity in **context of justification**
    - from Helmholtz to Poincaré: choice among $\underline{\text{different geometries}}$
- Bachelard (1929) $\Longrightarrow$ general relativity in **context of discovery**
    - from Riemann to tensor calculus: $\underline{\text{same geometry}}$ in different coordinate systems

Introduction

General relativity requires that fundamental laws be expressed as tensor equations:

- **comma-goes-to-semicolon rule**: ordinary derivatives (commas) $\Longrightarrow$ covariant derivatives (semicolons).
- **equivalence principle**: special-relativistic non-gravitational laws $\Longrightarrow$ general-relativistic gravitational laws

$$\overbrace{F^{\mu\nu}{}_{\underbrace{,\nu}_{\text{ordinary derivative}}} = -\frac{4\pi}{c} j^{\mu}}^{\text{non-gravitational law}} \quad \Longrightarrow \quad \overbrace{F^{\mu\nu}{}_{\underbrace{;\nu}_{\text{covariant derivative}}} = -\frac{4\pi}{c} J^{\mu}}^{\text{gravitational law}}$$

Introduction

- In 1929, when Bachelard’s book was published, the Vienna Circle manifesto advocated for “**neatness and clarity**” in German-speaking philosophy of science (Hahn, Neurath, and Carnap 1929, 15).
- Bachelard’s work as example of the growing fascination of French-speaking philosophy of science with the “**dark distances and unfathomable depths**” that the Viennese rejected (Hahn, Neurath, and Carnap 1929, 15).

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appreciate the ‘depth’ of Bachelard’s result by presenting it
with some of the ‘clarity’ that the logical empiricists aspired to

Part I

The Early Reception of Relativity in France

The Early Reception of Relativity in France

April 6, 1922: Einstein at the *Société française de philosophie*.

- **philosophers**:
  - Bergson, *Durée et simultanéité* (1922)
  - Brunschvicg, *L’expérience humaine et la causalité physique* (1922)
- **physicists**:
  - Becquerel, *Le principe de relativité et la théorie de la gravitation* (1922)
  - Metz, *La relativité* (1923)

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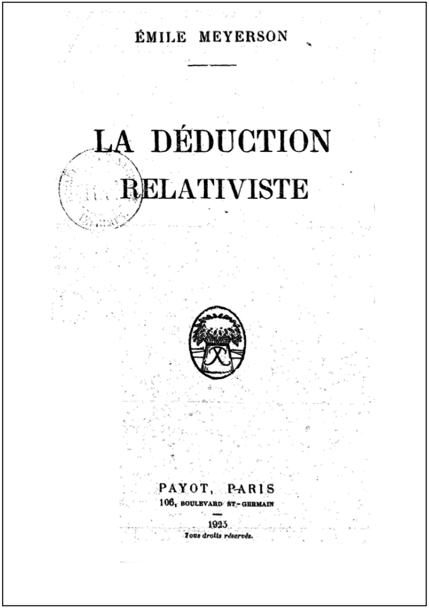
Meyerson, *La déduction relativiste* (1925)



ÉMILE MEYERSON

LA DÉDUCTION RELATIVISTE

PAYOT, PARIS
106, BOULEVARD ST-GERMAIN

1925
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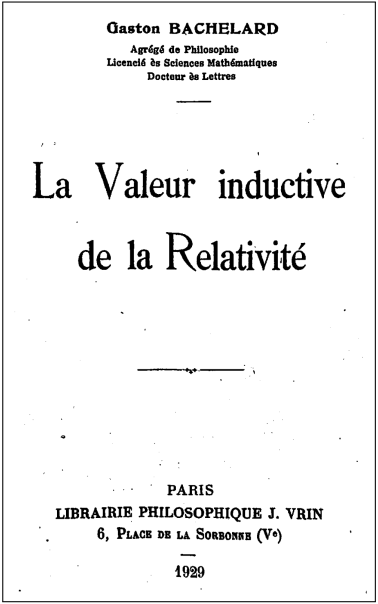

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The Early Reception of Relativity in France

- Meyerson: relativity as a **static deductive system** of existing laws;
- Bachelard: relativity as “a **method of progressive discovery**” of new laws (Bachelard 1929, 6).

inductive=heuristic

The Early Reception of Relativity in France

- general relativity by demanding that empirically well-established laws satisfy some **abstract mathematical requirements**, “finally drags the experience out of its initial domain of examination” (Bachelard 1929, 11).
- in general relativity putting a non-gravitational law in *tensor form* is sufficient to obtain a gravitational law, that is **new physical result**

⬇

mathematical induction

$$\text{non-gravitational law} \Longrightarrow \underline{\text{substitute ordinary with covariant derivatives}} \Longrightarrow \text{gravitational law}$$

The Early Reception of Relativity in France

> First of all, tensor calculus, which plays a primordial role in relativity, systematically pursues the maximum possible richness of variables. Through the interplay of its multiple indices, it is ready to face all instances of variation. On the other hand, the various tensorial indices fold and unfold at will in an alternating movement of generalization and application. [. . .] [T]hrough its condensed formulas, tensor calculus manages to inscribe generality under the persuasive sign of the particular. [. . .] Then, thanks to the simple movement of a coordinate transformation, it will be noticed that the experimental matter begins to flow into these formal molds.

(Bachelard 1929, 63)

Part II

The Mathematical Meaning of the Notion of Covariant Derivative

The Mathematical Meaning of the Notion of Covariant Derivative

$$A'_\sigma \frac{\partial \varphi}{\partial x'_\nu} = \underbrace{\frac{\partial x_\mu}{\partial x'_\nu}}_{\text{transformation coefficient}} \overbrace{\frac{\partial \varphi}{\partial x_\mu}}^{\text{derivative of a scalar } \varphi} \overbrace{A_\nu}^{\text{covariant vector (1-tensor)}} \, ,$$

$$\frac{\partial A'_\mu}{\partial x'_\nu} = \frac{\partial}{\partial x'_\nu} \left(\frac{\partial x_\sigma}{\partial x'_\mu} A_\sigma \right) = \underbrace{\frac{\partial x_\sigma}{\partial x'_\mu} \frac{\partial x_\tau}{\partial x'_\nu} \frac{\partial A_\sigma}{\partial x_\tau}}_{\text{tensorial term}} + \underbrace{\frac{\partial^2 x_\sigma}{\partial x'_\mu \partial x'_\nu} A_\sigma}_{\text{non-tensorial term}} \, .$$

The Mathematical Meaning of the Notion of Covariant Derivative

$$\underbrace{\left\{\begin{matrix}\mu\nu\\ \tau\end{matrix}\right\}}_{\text{Christoffel symbols}} = \frac{1}{2} g^{\lambda\alpha}\left(\overbrace{\frac{\partial g_{\mu\alpha}}{\partial x_\nu}}^{\text{metric tensor} = g_{\mu\nu}} + \frac{\partial g_{\nu\alpha}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right).$$

$$\left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\}' \frac{\partial x_\epsilon}{\partial x'_\rho} = \underbrace{\frac{\partial^2 x_\epsilon}{\partial x'_\mu \partial x'_\nu}}_{\text{non-tensorial term}} + \frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\beta}{\partial x'_\nu}\left\{\begin{matrix}\alpha\beta\\ \epsilon\end{matrix}\right\}.$$

The Mathematical Meaning of the Notion of Covariant Derivative

$$\frac{d^2 x_\epsilon}{\partial x'_\mu \partial x'_\nu} = \left\{ \begin{matrix} \mu\nu \\ \rho \end{matrix} \right\}' \frac{\partial x_\epsilon}{\partial x'_\rho} - \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\alpha}{\partial x'_\nu} \left\{ \begin{matrix} \alpha\beta \\ \epsilon \end{matrix} \right\}$$

$$\begin{aligned} \frac{\partial A'_\mu}{\partial x'_\nu} - \left\{ \begin{matrix} \mu\nu \\ \rho \end{matrix} \right\}' \overbrace{A_\sigma \frac{\partial x_\sigma}{\partial x'_\rho}}^{\hookrightarrow\, = A'_\rho} &= \frac{\partial x_\sigma}{\partial x'_\mu} \frac{\partial x_\tau}{\partial x'_\nu} \frac{\partial A_\sigma}{\partial x_\tau} + \frac{d^2 x_\epsilon}{\partial x'_\mu \partial x'_\nu} - \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} \left\{ \begin{matrix} \alpha\beta \\ \sigma \end{matrix} \right\} A_\sigma \\ &= \frac{\partial x_\sigma}{\partial x'_\mu} \frac{\partial x_\tau}{\partial x'_\nu} \underbrace{\left(\frac{\partial A_\sigma}{\partial x_\tau} - \left\{ \begin{matrix} \rho\sigma \\ \tau \end{matrix} \right\} A_\sigma \right)}_{\hookrightarrow\, \text{covariant 2-tensor}}, \end{aligned}$$

The Mathematical Meaning of the Notion of Covariant Derivative

$$\overbrace{A_{\mu\nu}}^{\text{covariant 2-tensor}} = \overbrace{\underbrace{\frac{\partial A_\mu}{\partial x_\nu}}_{\text{apparent change}} \underbrace{- \left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\} A_\rho}_{\text{pseudo-variation}}}^{\text{absolute change}} \; .$$

The Mathematical Meaning of the Notion of Covariant Derivative

> “It is known that the notion of the tensor derivative became established when there was a desire to find an expression that possessed the characteristics of a tensor and could replace, in all its roles, the ordinary derivative. It was recognized that the simple derivative of a vector does not maintain its form in any change of coordinates; in other words, it seemed that the ordinary derivative of a vector did not have the tensorial character. To align with the general spirit of tensor methods, it was necessary to add to the ordinary derivative terms capable of automatically compensating for the changes that this derivative underwent in a transformation of axes. This was easily achieved by adding linear functions of the Christoffel symbols”

(Bachelard 1929, 66 sq.)

The Mathematical Meaning of the Notion of Covariant Derivative

The addition of ‘ghost’ non-tensorial quantities, $\begin{Bmatrix} \mu\nu \\ \tau \end{Bmatrix}$, ensures the tensorial nature of the operation of tensor differentiation:

- $\begin{Bmatrix} \mu\nu \\ \tau \end{Bmatrix} = 0$ by a suitable choice of coordinates
  - “the covariant derivatives of tensors $\underline{\text{reduce}}$ to the ordinary derivatives” (Bachelard 1929, 66 sq.)
- $\begin{Bmatrix} \mu\nu \\ \tau \end{Bmatrix} \neq 0$ by *any* choice of coordinates
  - the covariant derivatives of tensors $\underline{\text{do not reduce}}$ to the ordinary derivatives

The Mathematical Meaning of the Notion of Covariant Derivative

$$\overbrace{R^{\rho}_{\mu\nu\sigma}}^{\text{Riemann-Christoffel Tensor}} = \overbrace{\begin{Bmatrix}\mu\sigma\\ \epsilon\end{Bmatrix}\begin{Bmatrix}\epsilon\nu\\ \rho\end{Bmatrix} - \begin{Bmatrix}\mu\nu\\ \epsilon\end{Bmatrix}\begin{Bmatrix}\epsilon\sigma\\ \rho\end{Bmatrix}}^{\text{terms depending on the first derivatives of the } g_{\mu\nu}} + \underbrace{\frac{\partial}{\partial x_{\nu}}\begin{Bmatrix}\mu\sigma\\ \rho\end{Bmatrix} - \frac{\partial}{\partial x_{\sigma}}\begin{Bmatrix}\mu\nu\\ \rho\end{Bmatrix}}_{\text{terms depending on second derivatives of the } g_{\mu\nu}},$$

- $R^{\rho}_{\mu\nu\sigma} = 0$: the non-vanishing of the Christoffel symbols due to an arbitrary choice of coordinates $\Longrightarrow$ **Euclidean**
- $R^{\rho}_{\mu\nu\sigma} \neq 0$: the non-vanishing of the Christoffel symbols are *not* due to an arbitrary choice of coordinates $\Longrightarrow$ **non-Euclidean**

Part III

The Physical Meaning of the Notion of Covariant Derivative

The Physical Meaning of the Notion of Covariant Derivative

Inductive value of the procedure “that allows us to replace, in certain cases, the **ordinary derivative with the tensor derivative**” (Bachelard 1929, 65).

- express non-gravitational physical law in orthogonal coordinates in terms of ordinary derivatives in the Euclidean case $R^{\rho}_{\mu\nu\sigma} = 0$
- replace ordinary derivatives with covariant derivatives in the Euclidean case $R^{\rho}_{\mu\nu\sigma} = 0$.
- postulate that the law in tensor applies in a non-Euclidean case $R^{\rho}_{\mu\nu\sigma} \neq 0$

non-gravitational laws ⟹ gravitational laws

The Physical Meaning of the Notion of Covariant Derivative

> “How can we not see that these few lines contain the essence of an extremely new method, which bases all its justification on a generalizing relation, and all its movement on an inductive impulse! There are three moments in this method:
>
> 1. Purely formal additions that contribute absolutely nothing in terms of quantity;
> 2. An algebraic game [*jeu algébrique*] that allows one to move from a particular case to the general case;
> 3. Then, once generality is conquered, a statement that invariance does not apply to a world of ghosts, but that almost always, thanks to the consistency and permanence of its form, this invariance implies a matter. Moreover, in Einstein’s principle of equivalence, one find assurance for this adventurous induction which claims, through a form, to conquer a matter.”

(Bachelard 1929, 66 sq.)

The Physical Meaning of the Notion of Covariant Derivative

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ct\,,$$

wave equation

$$\underbrace{\Box}_{\text{d'Alembert operator}}\overbrace{\varphi}^{\text{potential}} = -\frac{\partial^2\varphi}{\partial x_1^2} - \frac{\partial^2\varphi}{\partial x_2^2} - \frac{\partial^2\varphi}{\partial x_3^2} + \frac{\partial^2\varphi}{\partial x_4^2} = 0\,,$$

$$\underbrace{g^{11} = g^{22} = g^{33} = -1; \quad g^{44} = +1\,;}_{\text{orthogonal coordinates}} \quad \left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\} = 0; \quad R^{\rho}_{\mu\nu\sigma} = 0\,.$$

The Physical Meaning of the Notion of Covariant Derivative

$$g^{\mu\nu}\overbrace{\varphi_{\mu\nu}}^{\text{covariant 2-tensor}} \equiv g^{\mu\nu}\left(\underbrace{\frac{\partial^2\varphi}{\partial x_\mu \partial x_\nu}}_{\text{not a tensor}} - \underbrace{\begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}\frac{\partial\varphi}{\partial x_\alpha}}_{\text{not a tensor}}\right) = 0\,.$$

The Physical Meaning of the Notion of Covariant Derivative

$$\Box\varphi = 0 \quad \Longrightarrow \quad g^{\mu\nu}\varphi_{\mu\nu} = 0\,,$$

wave equation in all coordinate system
+ wave equation in a gravitational field

The Physical Meaning of the Notion of Covariant Derivative

> We have thus found the differential equation, correct and complete, capable of determining the law of the propagation of the potential φ in the case where this propagation occurs through a gravitational field. In this way, algebra has been induced to cooperate with reality, with its own impulse towards calculation, without ever assuming and seeking instruction from reality as primary. To summarize, let's take an overall look at the stages of the construction. The problem was approached through its formal characteristics. Then the tensorial character, which was truly mutilated by the degeneration of certain variations, was sought. Once highlighted, this tensorial element, by itself, restored the law in its entirety. An invariant character then presented itself, allowing the transition from the particular case to the general case. Finally, the assertion of the principle of equivalence regulated the osculation of reality through the laboriously and progressively constructed general framework

The Physical Meaning of the Notion of Covariant Derivative

1. equation expressing a **non-gravitational law** in which only ordinary derivatives appear: $g_{\mu\nu} =$ const., $\left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\} = 0$, $R^{\rho}_{\mu\nu\sigma} = 0$;
2. equation in **tensor form** by substituting ordinary derivatives with covariant derivatives: $g_{\mu\nu} \neq$ const., $\left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\} \neq 0$, $R^{\rho}_{\mu\nu\sigma} = 0$;
3. equation expressing a **gravitational law**: $g_{\mu\nu} \neq$ const. $\left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\} \neq 0$, $R^{\rho}_{\mu\nu\sigma} \neq 0$;

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equivalence principle: the gravitational laws depend only on $\left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\}$, but not on $R^{\rho}_{\mu\nu\sigma}$

Comma-Goes-to-Semicolon

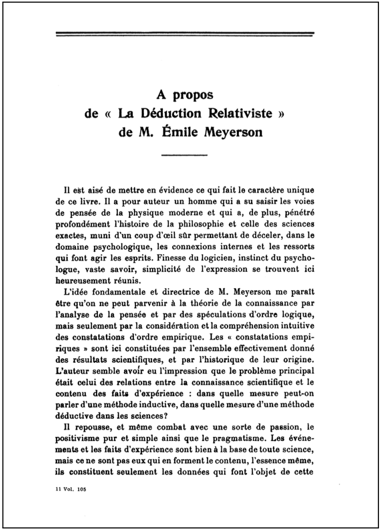
$$\overbrace{F^{\mu\nu}{}_{\underbrace{,\nu}_{\text{ordinary derivative}}} = -\frac{4\pi}{c} j^{\mu}}^{\text{non-gravitational law}} \Longrightarrow \overbrace{F^{\mu\nu}{}_{\underbrace{;\nu}_{\text{covariant derivative}}} = -\frac{4\pi}{c} J^{\mu}}^{\text{gravitational law}}$$

$$F^{\mu\nu}_{,;\nu} = \partial_{\nu} F^{\mu\nu} + \underbrace{\Gamma^{\mu}_{\nu\rho}}_{\text{affine connection=Christoffel symbols}} F^{\nu\rho}$$

Part IV

Conclusion

Conclusion



A propos
de « La Déduction Relativiste »
de M. Émile Meyerson

Il est aisé de mettre en évidence ce qui fait le caractère unique de ce livre. Il a pour auteur un homme qui a su saisir les voies de pensée de la physique moderne et qui a, de plus, pénétré profondément l'histoire de la philosophie et celle des sciences exactes, muni d'un coup d'œil sûr permettant de déceler, dans le domaine psychologique, les connexions internes et les ressorts qui font agir les esprits. Finesse du logicien, instinct du psychologue, vaste savoir, simplicité de l'expression se trouvent ici heureusement réunis.

L'idée fondamentale et directrice de M. Meyerson me paraît être qu'on ne peut parvenir à la théorie de la connaissance par l'analyse de la pensée et par des spéculations d'ordre logique, mais seulement par la considération et la compréhension intuitive des constatations d'ordre empirique. Les « constatations empiriques » sont ici constituées par l'ensemble effectivement donné des résultats scientifiques, et par l'historique de leur origine. L'auteur semble avoir eu l'impression que le problème principal était celui des relations entre la connaissance scientifique et le contenu des faits d'expérience : dans quelle mesure peut-on parler d'une méthode inductive, dans quelle mesure d'une méthode déductive dans les sciences?

Il repousse, et même combat avec une sorte de passion, le positivisme pur et simple ainsi que le pragmatisme. Les événements et les faits d'expérience sont bien à la base de toute science, mais ce ne sont pas eux qui en forment le contenu, l'essence même, ils constituent seulement les données qui font l'objet de cette

11 Vol. 105

Einstein's 1928 Review of Meyerson's Book

- **Meyerson** = deduction
- **Bachelard** = induction

> Il y a quelques années, M. Langevin nous disait: "Le Calcul Tensoriel sait mieux la physique que le Physicien lui-même"
>
> *(Bachelard 1934, 54)*

> Tensor calculus, Paul Langevin liked to say, knows relativity better

Conclusion

> “one obtains tensors again through differentiation of tensors with respect to the coordinates in an inertial system[*] and that e.g. the wave equation represents an objective expression in inertial system. The [affine connection Γ^{l}_{ik}]s now allows such tensor formation by differentiation in relation to an arbitrary coordinate system[†]. Therefore it is the invariant substitute of inertial systems and thereby—as it appears—the foundation of every relativistic field theory.”

Einstein to Besso, 10-08-1954

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[*] ordinary derivative.

[†] covariant derivative.

Thanks!

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