‘The Tensor Calculus Knows Physics Better Than the Physicist’: Bachelard on the Role of ‘Covariant Differentiation’ in Relativity Theory

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Gaston Bachelard’s 1929 *La valeur inductive de la relativité* is arguably his most overlooked work. This paper argues that, nevertheless, it represents a noteworthy contribution to the early history of the philosophical interpretation of general relativity. In particular, Bachelard deserves recognition for identifying the central importance of the comma-goes-to-semicolon rule, referenced in modern textbooks: The principle of general relativity demands that fundamental laws of nature be expressed by tensor equations, substituting ordinary derivatives (commas) with covariant derivatives (semicolons). The equivalence principle might be taken as the claim that this formal change in non-gravitational laws directly leads to the discovery of gravitational laws. The paper concludes that, among early professional philosophers working on relativity, Bachelard uniquely grasped the “inductive value” of this heuristic device: “tensor calculus knows physics better than the physicist does”, as Langevin once put it.

Introduction

Gaston Bachelard’s 1929 *La valeur inductive de la relativité* remains to this day the least cited and discussed among his works, and it is barely known outside Bachelard scholarship (Alunni 1999, 2019). As its title not so subtly insinuates, the booklet was written to counter Émile Meyerson’s 1925 *La déduction relativiste* (Meyerson 1925; see Parrochia 2003; Chazal 2012). However, it was published only a few months after Einstein’s (1928) glowing review of the latter appeared in the *Revue philosophique* (Giovanelli 2018). Einstein’s apparent endorsement of Meyerson’s ‘deductivist’ reading of relativity likely contributed to diminishing the prospects of Bachelard’s alternative ‘inductivist’ interpretation. The shift in the philosophical debate towards new quantum mechanics probably did the rest: *La valeur inductive* received only a few reviews and soon fell into oblivion. Bachelard himself hardly mentioned relativity in the following years (Fruteau de Laclos 2005). Reportedly, Bachelard’s daughter Suzanne and Georges Canguilhem, Bachelard’s successor at the Sorbonne, vetoed its reissue, arguing that Bachelard himself would have disowned it (Parrocchia 2014, 7-8; fn. 1).

In spite of these ‘prohibitions,’ a reprinting of *La valeur inductive* was published by Vrin in 2014. At a time of renewed interest in the history of French ‘epistemology’ (Brenner and Gayon 2009; Bitbol and Gayon 2015), the reedition did not fail to provoke reactions among Bachelard's scholars. In particular, Daniel Parrocchia (2014)'s illuminating 'Preface' has reopened the question of whether Bachelard's 'silence' on relativity starting from the mid-thirties (Fruteau de Laclos 2005) is a symptom of a change in his philosophy or merely the consequence of the declining interest in relativity[1] after the mid-1920s (Alunni 2019). However, the reedition of the booklet does not seem to have attracted much attention outside the circle of French Bachelard scholars (Abramo 2019). Indeed, although there has been a flourishing of historical work on the early philosophical reception of relativity, Bachelard's book is rarely mentioned in this literature.[2]

The lack of popularity of *La valeur inductive* is, after all, not surprising. Bachelard's arguments are expressed in convoluted language that might be dismissed as unnecessarily abstruse even by philosophically minded physicists. At the same time, reading the book presupposes some familiarity with the mathematical apparatus of general relativity, which might be quite demanding even for scientifically minded philosophers. This is a pity, since, as this article aims to show, Bachelard made a valuable contribution to the early history of the philosophical reception of general relativity that has remained largely unnoticed.

In the early philosophical literature on relativity theory, from Moritz Schlick (1917) to Hans Reichenbach (1928), it was common to project the theory against the background of the 19th-century debate about the foundations of geometry, from Helmholtz (1868) to Poincaré (1891). As a consequence, general relativity theory was often presented as a case of 'mathematical underdetermination': experience does not provide enough mathematical structure, so the choice among *different* (Euclidean and non-Euclidean) geometries is seen as arbitrary or conventional. In contrast, Bachelard recognized that general relativity emerged from a different tradition that, from Riemann's (1854–68) inaugural lecture, led to Levi-Civita and Ricci-Curbastro's (1900) absolute differential calculus—the 'tensor calculus' in modern terms. Thus, he perceived general relativity as a case of 'mathematical overdetermination': the theory exploits the fact that tensor calculus allows for the presentation of the *same* geometry in different, arbitrary coordinate systems.

Bachelard was the first and possibly the only philosopher to notice that it is by exploiting this 'redundancy' in the *description* of the real that general relativity, quite paradoxically, claims to uncover a *property* of the real itself.[3] The originality of Bachelard's perspective is a consequence of his distinct approach to the philosophy of physics. Most early philosophical debate considered general relativity within the 'context of justification,' addressing the question of how its abstract mathematical apparatus is connected to empirical reality. In contrast, Bachelard placed the theory in the 'context of discovery,' aiming to understand the role played by the mathematical apparatus in uncovering new physical content.[4] It is for this reason that Bachelard understood quite clearly the central role played by the procedure that modern textbooks call the comma-goes-to-semicolon rule (Misner, Thorne, and Wheeler 1973, 387–392). The principle of general relativity demands that all fundamental laws of nature be

[1] See Eisenstaedt 1986.

[2] Hentschel (1990), which remains the most comprehensive source on the topic, mentions Bachelard only briefly (71). Ryckman 2005, sets the cut-off at 1925. However, Ryckman's (2024) SEP entry on the early interpretation of general relativity also does not mention Bachelard.

[3] Sciama (1964) labeled this procedure the 'gauge trick'.

[4] Cassirer (1921) also insists on the 'heuristic value' of both relativity principles. However, Bachelard does not seem to be familiar with Cassirer's work.

specified by tensor equations; in practice, this requires substituting ordinary derivatives ('commas' in modern notation) with covariant derivatives ('semicolons').[5] The claim that this mere mathematical step leads to a physical result might be taken as a formulation of the 'equivalence principle'[6]: a trivial alteration in the mathematical formulation of special-relativistic non-gravitational laws leads to the non-trivial discovery of the corresponding gravitational laws that, at least in principle, can be tested empirically. As Bachelard loved to put it, quoting Paul Langevin: "tensor calculus knows physics better than the physicist does" (Bachelard 1934, 54; 1949, 578).

Bachelard should be credited for having at least attempted to provide a first 'philosophical' appreciation of tensor calculus,[7] by emphasizing the 'inductive value' of the 'trick' of replacing ordinary with covariant derivatives. Unfortunately, Bachelard's result remained obscured by the dense prose that characterizes *La valeur inductive.* In 1929, when Bachelard's book was published, the Vienna Circle manifesto advocated for "neatness and clarity" in German-speaking philosophy of science (Hahn, Neurath, and Carnap 1929, 15). One might say that Bachelard's work offers an example of the growing fascination of French-speaking philosophy of science with the "dark distances and unfathomable depths" that the Viennese rejected (15). The main objective of this paper is to appreciate the 'depth' of Bachelard's result by presenting it with some of the 'clarity' that the logical empiricists aspired to. To this purpose, after briefly placing Bachelard's book in its historical context (section 1), the paper analyzes in some detail the central example of 'relativistic induction' provided by Bachelard in the second chapter of the book: it presents the mathematical notion of covariant derivative (section 2) and its physical application (section 3).

Bachelard was confident that it was possible to appreciate the role of tensor calculus in relativity "without the need to understand the meaning of the symbols" (Bachelard 1929, 77). However, the poor reception of the book might suggest that he was too optimistic. This paper is based on the conviction that, in order to properly appreciate Bachelard's argument, it is indeed advantageous to become somewhat familiar with the formalism to which Bachelard alludes, without presenting it. Bachelard's knowledge of the mathematical apparatus of relativity is mostly based on Jean Becquerel's 1922 relativity textbook, the first authored by a French physicist (Becquerel 1922a, see also). In turn, the latter heavily draws on Arthur S. Eddington's *Espace, Temps et Gravitation* (Eddington 1921), the French translation of his celebrated popular book (Eddington 1920b), which was supplemented by a second detailed technical part.

A historical understanding of Bachelard's sources is important for appreciating Bachelard's philosophical point. In particular, I argue that Eddington's distinction between an 'artificial' and 'permanent' gravitational field, through book, percolated into Bachelard's work, forming the backbone of his argument. Bachelard recognized that the "inductive audacity" of the theory (Bachelard 1929, 74) resides in the following requirement: All laws describing phenomena in an 'artificial' gravitational field that can be removed by the choice of coordinates will also hold in a permanent gravitational field (Eddington 1920a, 43). This requirement, which might be taken as a formulation

[5]The ordinary derivative of a contravariant vector is often indicated by $A^{\mu}_{,\nu} = \partial_{\mu} A_{\nu}$, the covariant derivative by $A^{\mu}_{;\nu} = \nabla_{\mu} A_{\nu}$. See below, fn. 20, for an example. In the following, the notation used at Bachelard's time is adopted.

[6]The formulation of the equivalence principle remains controversial (see Lehmkuhl 2022).

[7]It is indeed surprising that even Reichenbach (1928) barely addresses the meaning of tensor calculus for general relativity.

of the equivalence principle, cannot be 'deductively' proved; its justification resides in the fact that it is 'inductively' powerful in delivering new gravitational laws.

1 Bachelard and the Early Reception of Relativity in France

One could argue that the history of the philosophical reception of relativity in France has a precise start date: April 6, 1922, the day of the legendary session organized by the *Société française de philosophie* at the Sorbonne on the occasion of Einstein's visit to Paris (Einstein et al. 1922). Among the participants were leading French mathematicians, such as Élie Cartan; physicists, like Langevin and Becquerel; and philosophers, like Henri Bergson, Léon Brunschvicg, and Émile Meyerson. A few months after Einstein's visit, in the spring, the publisher Alcan released *Durée et simultanéité* by Bergson (1922), and in the summer, *L'expérience humaine et la causalité physique* by Brunschvicg (1922), which also includes an analysis of both relativity theories (401–432). In the same year, Becquerel[8] published two books on relativity: a popular book that provides a general exposition intended for a non-specialist audience (Becquerel 1922a) and a more technical book, dedicated to Langevin—one of the early supporters of the theory of relativity in France (Becquerel 1922b)—, which compiles Becquerel's 1921–1922 Paris lectures (Imperiali 2024, chap. IV).

Becquerel sought not only to spread the "new ideas" but also to defend them against misconceptions and criticisms (Becquerel 1922a, 9). In 1923, after releasing a scathing review of Bergson's book (Becquerel 1923), Becquerel wrote the 'Preface' to *La relativité* by André Metz (1923), which undertook a systematic 'debunking' of French vulgarizations and supposed refutations of the theory (Einstein to Metz, Dec. 25, 1923; CPAE, Vol. 14, Abs. 234). The theory of relativity is a physical theory that explains 'real phenomena' and has nothing to do with 'philosophical relativity' —a point that both Bergson (Metz 1924, 81) and Brunschvicg misunderstood (Metz 1926, 71). It was Meyerson who was able to articulate this view more broadly in a monograph that was sent to the publisher in March 1924. Meyerson's *La déduction relativiste* appeared the following year and sparked considerable debate (Lalange 1925; Brunschvicg 1926; Metz 1927). After a few months of epistolary back and forth, Einstein (1927) himself completed a long review of Meyerson's book, promoting it as a realistic-rationalistic antidote to the widespread phenomenistic-positivistic interpretation of the theory. This review, translated by Metz, was ready for publication in early December 1927 (Giovanelli 2018).

At that time, the 43-year-old Bachelard was still a high school teacher of physics and chemistry in his hometown, Bar-sur-Aube (Chimisso 2001, 1; Smith 2016, 5). At the end of November, he contacted Meyerson to introduce himself and inform him of the shipment, by the Parisian publisher Vrin, of two of his early philosophical writings (Bachelard to Meyerson, Nov. 30, 1927; EMLF, 32): his doctoral thesis supervised by Brunschvicg (Bachelard 1928a) along with the complementary dissertation (Bachelard 1928b). Bachelard visited Meyerson in Paris in the spring of 1928 (Bachelard to Meyerson, Apr. 18, 1929; EMLF, 32), around the time Einstein's (1928) review appeared in the *Revue philosophique*. Bachelard aimed to discuss his plans to work on relativity.

Indeed, a few months later, he sent a copy of *La valeur inductive* to Meyerson, emphasizing that it was only "the outline of a thought that I would like to develop in a more extensive work" (Bachelard to Meyerson, Dec. 18, 1927; EMLF, 32f.). Bachelard's

[8]Son of Nobel Prize-winning physicist Henri Becquerel.

choice to consult Meyerson is understandable, given that Meyerson had become the leading French philosopher in the field. However, Bachelard's book was written with an unmistakable anti-Meyersonian bent, which was conspicuously signaled by the title itself. Indeed, Bachelard's use of the term 'inductive value' is somewhat idiomatic and serves as a counterpoint to Meyerson's insistence on the 'deductive value' of the theory. I think that the expression 'heuristic value'[9] would have been less catchy but probably more appropriate to convey the central idea of the book.

According to Bachelard, relativity is not, as Meyerson suggested, a static deductive system of available laws; on the contrary, it is "a method of progressive discovery" of new laws (Bachelard 1929, 6). In particular, in the second chapter, Bachelard analyzes some examples that should introduce the reader to the "very heart of mathematical induction" that characterizes relativistic physics (10). Relativity (both in its special and general variants), by demanding that empirically well-established laws satisfy some abstract mathematical requirements, "finally drags the experience out of its initial domain of examination" (11). Here lies the distinctive character of the theory. Newtonian theory was ultimately an attempt at an *explanation* of the laws of nature according to a "deductive ideal" modeled on Euclidean geometry (51). On the contrary, according to Bachelard, Einstein's theory is a method of *discovery* of the laws of nature based on an 'inductive ideal' represented by thermodynamics (140f.).[10] According to Bachelard, this "richness of inference" (51) is not simply an occasional byproduct of the theory. On the contrary, it is its true driving force: "the inferential value is one of the deepest, and also one of the most curious, characteristics of Einsteinian thought" (52).

While Bachelard occasionally refers to special relativity, his main focus is general relativity. The latter is based on the requirement that the special relativistic laws of nature, valid only for a particular class of coordinate systems, should be put in 'tensor form' valid in all coordinate systems. At first glance, this might seem like a rather trivial change in their mathematical formulation[11]; however, combined with the equivalence principle, this change is nontrivial in its implications. It incorporates gravity into all the laws of physics, leading to the discovery of new testable gravitational laws starting from empirically confirmed non-gravitational laws. In this sense, Bachelard emphasizes that "[i]nduction, more here than elsewhere, is the very movement of the system; it is invention elevated to the status of a method" (51). Relativity arrives at reality indirectly by putting the laws of nature in their most general form; that is, to a certain extent, it "proves reality by generality" (52). This strategy can be framed as an inductive recipe that, for Bachelard, has profound philosophical importance: "what can be generalized, must be generalized; it is precisely this that will complete our knowledge of reality" (52).

The history of science presents many other cases of "mathematical generalizations" (55). Bachelard concedes that, at first glance, this fact "might serve as an argument for a deductive thesis" *à la* Meyerson (63). However, upon closer inspection, most of these cases should be regarded as "instances of induction" (63). For Bachelard, general relativity provides a prime example of this peculiar form of "mathematical induction"

[9]An expression that Einstein uses on several occasions, e.g., Einstein 1917, 28f.; 67.

[10]For the comparison with thermodynamics, Bachelard refers to Campbell 1924. Bachelard seems to be unaware that Einstein (1919) himself suggested this comparison.

[11]Bachelard seems to be unaware of Kretschmann's (1917) triviality-objection.

(51).[12] Relativity shows that expressing equations of various kinds in their most general mathematical tensor *form*, which holds in any coordinate system, directly leads to the discovery of new physical *content*:

> First of all, tensor calculus, which plays a primordial role in relativity, systematically pursues the maximum possible richness of variables. Through the interplay of its multiple indices, it is ready to face all instances of variation. On the other hand, the various tensorial indices fold and unfold at will in an alternating movement of generalization and application. The general always remains present, always as clear as the example. In other words, through its condensed formulas, tensor calculus manages to inscribe generality under the persuasive sign of the particular. Therefore, tensor calculus seems to us, in its essence, particularly suited to providing the framework for generalization. By giving it one of the variables of the problem, it will be able to associate all the others with it; it will prepare them as empty forms, as possibilities awakened by a sort of instinct for functional symmetry, by a genius of generality. Then, thanks to the simple movement of a coordinate transformation, it will be noticed that the experimental matter begins to flow into these formal molds, bringing life to these ghosts, balancing all variations, and finally elucidating the role of the general. (Bachelard 1929, 63)

Bachelard seems to be fascinated precisely by that 'debauch of the indices' (*débauches d'indices*) about which Cartan (1928, V) complained around the same time. Tensor calculus, through the skillful manipulation of index notation, allows one to generalize the mathematical expression of a law, which is valid in the special case of rectangular coordinates, to a form that is valid in all possible coordinate systems: "It is possible to grasp this surprising process of generalization at the very root of tensor calculus" (Bachelard 1929, 64). It is the formal manipulation itself that leads to new empirically testable results.

To make this point clear, in the second chapter of the book, Bachelard discusses in particular one example: the procedure of replacing ordinary derivatives with covariant derivatives in the theory's equations (65–71).[13] As we shall see, to this purpose relativists do not hesitate to introduce non-tensorial 'ghost quantities,' the so called, Christoffel symbols, that, in certain circumstances, can be made vanish by mere choice of the coordinates: "The ghost quantity [*quantité fantôme*] neither exists numerically nor formally. It is, above all, a pure nothingness. And it is naturally for this reason that it can be added to a numerical quantity" (64). Indeed, the introduction of such quantities is justified because of their 'inductive value,' as it reestablishes the full tensorial form of a law by compensating for the terms that transform non-tensorially under an arbitrary change of coordinates. According to Bachelard, the addition of 'ghost quantities' is not simply an occasional trick but a general methodological strategy: "Tensor calculus has made it a method. We will try to shed light on the spirit of this method" (65).

The objective of this paper is to elucidate the workings of this mathematical technique, which Bachelard only briefly references. This exercise is, I think, useful (a) to clarify Bachelard's arguments, which are often presented in a less than straightforward manner and (b) to check whether they stand up to closer scrutiny. To this end, the paper introduces a few mathematical notions that may be obvious to specialists but less accessible to the general reader. We hope to strike a balance by following Bachelard's main source, namely Becquerel's (1922a) relativity textbook—extensively

[12] Contrary to Abramo (2019, chap. 2), in my view, the expression 'mathematical induction' has little to do with the 'proofs by induction' used in mathematics.

[13] A second example, related to the discovery of the field equation (Bachelard 1929, 77–81) will be analyzed elsewhere.

quoted by Bachelard on several occasions. In turn, Becquerel 's work largely relies on Eddington's (1921) work, with which Bachelard was also familiar. Specifically, Eddington's distinction between 'artificial' and 'permanent gravitational' fields, as presented through Becquerel, plays an important role in Bachelard's book.

2 The Mathematical Meaning of the Notion of Covariant Derivative

Given the centrality that tensor calculus plays in Bachelard's book, it is quite surprising how little he attempts to present some basic notions of it to his readers. Bachelard simply alludes somewhat elusively to the presentation provided by Becquerel (1922a, 1922b). In particular, Bachelard presupposes that his readers are familiar with tensor transformation rules and the algebraic operations (addition, multiplication, and contraction) that can be performed on tensors at the same point in a manifold x_ν (for $\nu = 1 \ldots n$) (Becquerel 1922b, §63). His interest lies in the generalization of the notion of differentiation for tensors—the rate of change of tensor components when passing from the point x_ν to the neighboring point $x_\nu + dx_\nu$. Again, Bachelard could find a standard presentation of the problem in Becquerel's (1922b) book (§69). The derivative of a scalar $\frac{\partial \varphi}{\partial x_\nu}$, whose value varies from point to point, transforms like a covariant[14] vector A_ν under a change of coordinates $x'_\nu = f^\nu(x_\nu)$:

$$A'_\sigma = \underbrace{\frac{\partial x_\nu}{\partial x'_\sigma}}_{\text{transformation coefficient}} A_\nu\,, \tag{1}$$

where summation over all the values of the indices that appear twice—once as a subscript and once as a superscript—is implied.[15] One can then easily generalize these transformation rules to the case of quantities with an arbitrary number of components of covariant or contravariant character—the so-called *tensors*. One needs only to introduce partial derivatives of the coordinates, in one sense $\frac{\partial x'_\mu}{\partial x_\sigma}$ or the other $\frac{\partial x_\sigma}{\partial x'_\mu}$ respectively, for each contravariant A^μ and covariant component A_ν, and sum over repeated indices. While the numerical values of the components of a tensor change from one coordinate system to another, the tensorial machinery ensures that tensors are unaffected by coordinate transformations.

If the derivative of a scalar is, as we have seen, a tensor—specifically a covariant vector (which is a tensor of rank 1)—the derivative of a vector does not generally transform tensorially under coordinate transformations. This can be seen by carrying out the transformation explicitly. If we differentiate both sides of eq. 1 with respect to x'_ν, one obtains:

$$\frac{\partial A'_\mu}{\partial x'_\nu} = \frac{\partial}{\partial x'_\nu}\left(\frac{\partial x_\sigma}{\partial x'_\mu} A_\sigma\right) = \underbrace{\frac{\partial x_\sigma}{\partial x'_\mu}\frac{\partial x_\tau}{\partial x'_\nu}\frac{\partial A_\sigma}{\partial x_\tau}}_{\text{tensorial term}} + \underbrace{\frac{\partial^2 x_\sigma}{\partial x'_\mu \partial x'_\nu}}_{\text{non-tensorial term}} A_\sigma\,. \tag{2}$$

[14] By contrast, the velocity $\frac{dx_\nu}{dt}$ is an example of a contravariant vector since x_ν appears at the numerator.

[15] This is the so-called Einstein summation convention: $A_1 B^1 + A_2 B^2 + \ldots + A_n B^n = \sum_{\nu=1}^{n} A_\nu B^\nu = A_\nu B^\nu$.

where the last term disrupts the tensorial nature of the transformation. In the case of a linear transformation from one Cartesian coordinate system to another, the vector components at a neighboring point transform in the same way, that is, the transformation coefficients in eq. 1 are constant. However, if one performs a non-linear coordinate transformation and introduces curvilinear coordinates, like polar or cylindrical coordinates, the coefficients $\frac{\partial x_\sigma}{\partial x'_\mu}$ change with position. Thus, there is an extra term corresponding to their derivatives, which is generally $\neq 0$. As a consequence, if the derivative of a vector vanishes in one coordinate system (the vector field is uniform), it would become non-zero by simply changing to, say, polar coordinates (Becquerel 1922b, 165). It would then be impossible to write differential equations in tensorial form.

To overcome this difficulty, it is necessary to find a tensorial or covariant form of differentiation that replaces the ordinary derivative. To this end, it is necessary to introduce a new multi-component object (§69), the so-called Christoffel symbols of the first and second kinds:

$$\begin{bmatrix}\mu\nu \\ \lambda\end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right), \tag{3}$$

$$\begin{Bmatrix}\mu\nu \\ \tau\end{Bmatrix} = g^{\lambda\alpha}\begin{bmatrix}\mu\nu \\ \alpha\end{bmatrix} = \frac{1}{2}g^{\lambda\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x_\nu} + \frac{\partial g_{\nu\alpha}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right). \tag{4}$$

This 3-index symbol is a combination of (ordinary) derivatives of the $g_{\mu\nu}$'s; that is, it determines the rate of change of the $g_{\mu\nu}$'s with respect to the chosen coordinates. Thus, it vanishes if the $g_{\mu\nu}$'s are constant. The $g_{\mu\nu}$'s are the so-called 'fundamental tensor' and are defined as usual as

$$ds^2 = dx^\nu dx_\nu = g_{\mu\nu}dx^\mu dx^\nu = g^{\mu\nu}dx_\mu dx_\nu\,, \tag{5}$$

where $g^\nu_\mu = g_{\mu\sigma}g^{\nu\sigma} = 1$ or 0 depending on whether $\mu = \nu$ or $\mu \neq \nu$. Intuitively, the $g_{\mu\nu}$'s can be thought of as 'conversion factors' that take the coordinate difference between two neighboring points x_ν and $x_\nu + dx_\nu$ and produce their reciprocal distance ds. For this reason, the 'fundamental' tensor $g_{\mu\nu}$ is also called the 'metric,' that is, the 'measurement' tensor. In Cartesian or rectangular coordinates, the $g_{\mu\nu}$'s have constant values, and eq. 5 reduces to the sum of squares $ds^2 = dx^\mu dx^\mu$, which is nothing but the Pythagorean theorem. Under the change of coordinates, $\mathrm{d}x^\nu$ transform like a contravariant vector[16] and $g_{\mu\nu}$'s like a covariant tensor:

$$g'_{\mu\nu} = \frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\beta}{\partial x'_\nu}g_{\alpha\beta}\,. \tag{6}$$

One can then recover the same invariant distance ds between the corresponding points $x'_\nu + dx'_\nu$ in an arbitrary non-Cartesian coordinate system (such as polar or cylindrical coordinates).

The crucial point is that, unlike the $g_{\mu\nu}$'s, the Christoffel symbols *do not* generally transform as a tensor when switching to a different coordinate system. Indeed, if one differentiates the $g'_{\mu\nu}$ with respect to x'_λ, x'_μ, and x'_ν, with some manipulation, one can

[16] Nevertheless, for convenience, low indices are typically used for the coordinates.

obtain the transformation rules for the Christoffel symbols of the first kind:

$$\begin{bmatrix}\mu\nu\\ \lambda\end{bmatrix}' = g_{\alpha\beta}\frac{\partial^2 x_\alpha}{\partial x'_\mu \partial x'_\nu}\frac{\partial x_\beta}{\partial x'_\gamma} + \frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\beta}{\partial x'_\nu}\frac{\partial x_\gamma}{\partial x'_\lambda}\begin{bmatrix}\alpha\beta\\ \gamma\end{bmatrix}.$$

Multiplying both terms of $g'^{\lambda\rho}\frac{\partial x_\epsilon}{\partial x'_\rho}$, summing over λ and ρ, and simplifying, one obtains the transformation rules for the Christoffel symbols of the second kind:

$$\begin{Bmatrix}\mu\nu\\ \rho\end{Bmatrix}' \frac{\partial x_\epsilon}{\partial x'_\rho} = \underbrace{\frac{\partial^2 x_\epsilon}{\partial x'_\mu \partial x'_\nu}}_{\hookrightarrow \text{ non-tensorial term}} + \frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\beta}{\partial x'_\nu}\begin{Bmatrix}\alpha\beta\\ \epsilon\end{Bmatrix}. \tag{7}$$

As one can see, once again a term depending on the second derivatives of the coordinates appears, which spoils the tensorial nature of the Christoffel symbols: if they vanish in Cartesian coordinates where the $g_{\mu\nu}$'s are constant, they do not vanish in the general case where the $g_{\mu\nu}$'s are functions of the coordinates. In Bachelard's language, the Christoffel symbols are non-tensorial 'ghost quantities' that can be made to disappear by a suitable choice of coordinates. However, it is precisely for this reason that they can be used to 'compensate' for how the partial derivative fails to transform like a tensor. One just needs to take the definition of the non-tensorial term

$$\frac{d^2 x_\epsilon}{\partial x'_\mu \partial x'_\nu} = \begin{Bmatrix}\mu\nu\\ \rho\end{Bmatrix}' \frac{\partial x_\epsilon}{\partial x'_\rho} - \frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\alpha}{\partial x'_\nu}\begin{Bmatrix}\alpha\beta\\ \epsilon\end{Bmatrix}$$

from eq. 7 and substitutes in eq. 2:

$$\frac{\partial A'_\mu}{\partial x'_\nu} - \begin{Bmatrix}\mu\nu\\ \rho\end{Bmatrix}' \underbrace{A_\sigma \frac{\partial x_\sigma}{\partial x'_\rho}}_{\hookrightarrow = A'_\rho} = \frac{\partial x_\sigma}{\partial x'_\mu}\frac{\partial x_\tau}{\partial x'_\nu}\frac{\partial A_\sigma}{\partial x_\tau} - \frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\beta}{\partial x'_\nu}\begin{Bmatrix}\alpha\beta\\ \sigma\end{Bmatrix} A_\sigma = \underbrace{\frac{\partial x_\sigma}{\partial x'_\mu}\frac{\partial x_\tau}{\partial x'_\nu}\left(\frac{\partial A_\sigma}{\partial x_\tau} - \begin{Bmatrix}\rho\sigma\\ \tau\end{Bmatrix} A_\sigma\right)}_{\hookrightarrow \text{ tensor}},$$

where the last term is obtained by renaming the repeated indices α, β, σ as σ, τ, ρ,[17] and regrouping. As a result, one obtains the transformation law of a tensor. We can then replace the ordinary derivative with respect to rectangular coordinates with a covariant derivative that is independent of the choice of coordinates:

$$A_{\mu\nu} = \overbrace{\underbrace{\frac{\partial A_\mu}{\partial x_\nu}}_{\text{apparent change} \hookleftarrow} - \underbrace{\begin{Bmatrix}\mu\nu\\ \rho\end{Bmatrix} A_\rho}_{\hookrightarrow \text{ pseudo-variation}}}^{\text{absolute change}}. \tag{8}$$

As one can see, in order to produce the covariant derivative, the ordinary derivative $\frac{\partial A_\mu}{\partial x_\nu}$, that is, the *apparent* rate of the components of a vector in passing from x_ν to

[17] An index summed over is called a *indice muet* or 'dummy index' and can be renamed freely since it doesn't appear in the final result (Becquerel 1922b, 149f.).

$x_\nu + dx_\nu$ with respect to a given coordinate system, must be supplemented by another correction term, which reflects the "pseudo-variation" $-\begin{Bmatrix} \mu\nu \\ \rho \end{Bmatrix} A_\rho$ attributable to the curvilinearity of the coordinates (Becquerel 1922b, 171)—e.g., in the case one uses polar or cylindrical coordinates (see also Eddington 1921, 50–53). Thus, through the addition of a 'ghost' term that vanishes in orthogonal coordinates, eq. 8 determines the *absolute* change in the vector. While the two summands of eq. 8 do *not* transform like a tensor, eq. 8 as a whole is a tensor equation.

The covariant tensor $A_{\mu\nu}$ is called the covariant derivative of the covariant vector A_μ; if it vanishes in one coordinate system, it vanishes in all coordinate systems. In a somewhat similar manner, one can obtain formulas for the covariant derivatives of covariant, contravariant, and mixed tensors of any order (Becquerel 1922b, 166–168). As a rule of thumb, it can be noted that in all cases, differentiation adds a covariant index, that is, a subscript index. We can use Bachelard's somewhat elliptic formulation to summarize this result:

> It is known that the notion of the tensor derivative became established when there was a desire to find an expression that possessed the characteristics of a tensor and could replace, in all its roles, the ordinary derivative. It was recognized that the simple derivative of a vector does not maintain its form in any change of coordinates; in other words, it seemed that the ordinary derivative of a vector did not have the tensorial character. To align with the general spirit of tensor methods, it was necessary to add to the ordinary derivative terms capable of automatically compensating for the changes that this derivative underwent in a transformation of axes. This was easily achieved by adding linear functions of the Christoffel symbols. (Bachelard 1929, 66f.)

It is then the addition of 'ghost' non-tensorial quantities, the Christoffel symbols, that assures the tensorial nature of the operation of tensor differentiation. There are "simple cases" in which the Christoffel symbols are "identically zero", since they can be made to vanish by choosing Cartesian coordinates, where the $g_{\mu\nu}$'s are constant (66f.). This means that in Cartesian coordinates, "the covariant derivatives of tensors reduce to the ordinary derivatives" (66f.); however, it is not obvious that this choice is always possible.

In n dimensions, one is free to introduce new coordinates by providing n independent functions of the old coordinates. Thus, one can arbitrarily set n components of the $g_{\mu\nu}$'s as constant. However, a symmetric tensor ($g_{\mu\nu} = g_{\nu\mu}$) has $n(n+1)/2$ independent components. Thus, one is not free to do the same with the remaining $n(n-1)/2$ components. It is necessary to find a coordinate-independent criterion to establish when a given $g_{\mu\nu}$-system is reducible to a set of constant values (Becquerel 1922b, §73). By taking the first covariant derivative of the covariant tensor $g_{\mu\nu}$, one obtains a third-order covariant tensor $Q_{\mu\nu\sigma}$ which, however, vanishes identically. It is then necessary to arrive at a tensor of fourth order, depending on second-order covariant derivatives of the $g_{\mu\nu}$ This tensor is called the Riemann-Christoffel tensor and consists of first derivatives and squares of Christoffel symbols:

$$R^{\rho}_{\mu\nu\sigma} = \overbrace{\begin{Bmatrix}\mu\sigma\\ \epsilon\end{Bmatrix}\begin{Bmatrix}\epsilon\nu\\ \rho\end{Bmatrix} - \begin{Bmatrix}\mu\nu\\ \epsilon\end{Bmatrix}\begin{Bmatrix}\epsilon\sigma\\ \rho\end{Bmatrix}}^{\text{terms depending on the first derivatives of the } g_{\mu\nu}} + \underbrace{\frac{\partial}{\partial x_{\nu}}\begin{Bmatrix}\mu\sigma\\ \rho\end{Bmatrix} - \frac{\partial}{\partial x_{\sigma}}\begin{Bmatrix}\mu\nu\\ \rho\end{Bmatrix}}_{\text{terms depending on second derivatives of the } g_{\mu\nu}}, \tag{9}$$

thus, it depends only on the fundamental tensor $g_{\mu\nu}$. The vanishing of the Riemann-Christoffel tensor is the necessary condition that, by an appropriate choice of coordinates, ensures the metric coefficients assume constant values and Christoffel symbols vanish everywhere. A manifold that satisfies the condition $R^{\rho}_{\mu\nu\sigma} = 0$ is called 'Euclidean.' In this case, the variability of the $g_{\mu\nu}$'s and the non-vanishing of the Christoffel symbols are only due to an arbitrary choice of coordinates and can be 'transformed away' by reintroducing Cartesian coordinates. On the contrary, if $R^{\rho}_{\mu\nu\sigma} \neq 0$, the non-constancy of the $g_{\mu\nu}$'s is not only a manifestation of the choice of the coordinate system but an intrinsic property of the manifold. A manifold that satisfies this condition is called 'non-Euclidean.'

3 The Physical Meaning of the Notion of Covariant Derivative

Once we have outlined the notion of covariant derivative that Bachelard was probably familiar with, we are ready to appreciate his insistence on the inductive value of the procedure "that allows us to replace, in certain cases, the tensor derivative with the ordinary derivative" (Bachelard 1929, 65). Bachelard quotes a passage from Becquerel's book describing the process of generalization allowed by tensor calculus in its application to physics (Becquerel 1922b, 168). In a Euclidean universe where $R^{\rho}_{\mu\nu\sigma} = 0$, one starts from a physical law expressed in orthogonal coordinates by a relation that includes expressions that are clearly 'degenerate' forms of tensors and their ordinary derivatives. One can then proceed, still in orthogonal coordinates, by replacing the 'degenerate' forms with the tensors themselves, and ordinary derivatives with covariant derivatives. This is simply a formal reformulation of the same differential law in which the Christoffel symbols appear explicitly. The next step is to postulate that the same law applies in a non-Euclidean case $R^{\rho}_{\mu\nu\sigma} \neq 0$, when no rectangular coordinate system can be introduced and the Christoffel symbols cannot be made to vanish over an extended region (168). Bachelard writes the following comment:

> How can we not see that these few lines contain the essence of an extremely new method, which bases all its justification on a generalizing relation, and all its movement on an inductive impulse! There are three moments in this method:
>
> 1. Purely formal additions that contribute absolutely nothing in terms of quantity;
> 2. An algebraic game [*jeu algébrique*] that allows one to move from a particular case to the general case;
> 3. Then, once generality is conquered, a statement that invariance does not apply to a world of ghosts, but that almost always, thanks to the consistency and permanence of its form, this invariance implies a matter. Moreover, in Einstein's principle of equivalence, one find assurance for this adventurous induction which claims, through a form, to conquer a matter. (Bachelard 1929, 66f.)

The method Bachelard aims to describe is based on mere algebraic manipulation of the formulas. One replaces ordinary derivatives with covariant derivatives through purely 'formal addition,' that is, by introducing the 'pseudo-variation,' which is a linear function of the Christoffel symbols, eq. 8. In this way, the $g_{\mu\nu}$'s are allowed to be variable, and the expression of a law that was previously valid only in a special class of coordinates takes a general form that holds in all possible coordinate systems. The change is here purely 'formal' since $R^{\rho}_{\mu\nu\sigma} = 0$, and the non-constancy of the $g_{\mu\nu}$'s can be made to disappear by reintroducing Cartesian coordinates. The 'material change' is achieved by postulating that the law in its most general, tensorial form also applies to the general non-Euclidean case $R^{\rho}_{\mu\nu\sigma} \neq 0$ As Bachelard alludes to in the last part of the passage (and we shall see below), it is the equivalence principle that guarantees that the laws put in that form are those that hold in the presence of gravitation since they are supposed to depend only on the first derivatives of the $g_{\mu\nu}$.

If one accepts this assumption, by the purely formal procedure of bringing the special relativistic non-gravitational laws into tensor form, one can find the corresponding gravitational laws. Bachelard aims to elucidate "this *algebraic* induction by following its application to a specific example" (Bachelard 1929, 67). Bachelard's example, the only one he discusses in detail in the book, is taken from Becquerel (1922b, 168f.), who introduces it just after presenting the notion of the covariant derivative (§60). In turn, Becquerel sourced the example from Eddington (1921, 49f.). Bachelard's decision to highlight this specific case reveals his philosophical intentions. It's about applying the formalism outlined in the preceding section to a special case. As is well known, in special and general relativity, spacetime is a manifold x_μ with $\mu = 1, 2, 3, 4$, where

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ct\,,$$

The coordinates x_1, x_2, x_3 refer to spatial coordinates and $x_4 = ict$ (where c is the velocity of light) to the time coordinate; the distance between two close points is expressed by eq. 5, where ds is allowed to take negative values. Let's suppose we want to determine "the most general equation that governs the propagation of an arbitrary potential φ" (Bachelard 1929, 67) with the velocity of light. An equation of this kind is called a 'wave equation.' By setting $c = 1$, it becomes a second-order differential equation in the form:

$$\Box\varphi = -\frac{\partial^2\varphi}{\partial x_1^2} - \frac{\partial^2\varphi}{\partial x_2^2} - \frac{\partial^2\varphi}{\partial x_3^2} + \frac{\partial^2\varphi}{\partial x_4^2} = 0\,, \tag{10}$$

where $\Box$ is the so-called D'Alembert operator, *i.e.*, the four-dimensional Laplace operator ∇^2 (the divergence of the gradient), which makes the role of time variable in the propagation of waves explicit (68): "It is with this formula that the problem of setting up an equation for propagation in pre-relativistic mathematical physics is completed", that is, for the case $R^{\rho}_{\mu\nu\sigma} = 0$ (68). The examination of the coefficients of the four terms of the D'Alembertian $\Box\varphi$ shows that this equation holds with respect to a coordinate system where the components of the metric tensor, for $\mu = \nu$, take the 'degenerate' values:

$$g^{11} = g^{22} = g^{33} = -1; \quad g^{44} = +1\,,$$

while for $\mu \neq \nu$ they vanish. In a space-time setting, a coordinate system satisfying this condition might be called an inertial or Galileian coordinate system. This choice of coordinates implies that "the disappearance of the other second derivatives of φ results from their multiplication with the corresponding $g^{\mu\nu}$ which are zero" (68).

By using the formal tools provided by tensor calculus, one can rewrite expression eq. 10 to "restore all the second derivatives of φ and assign the derivatives taken with respect to two different variables the corresponding coefficient $g^{\mu\nu}$" (68). In other terms, we make generic $g^{\mu\nu}$ appear explicitly in the equation and substitute ordinary derivatives with covariant derivatives. Since the potential φ is a scalar, its ordinary derivative is, as we have seen, a covariant vector $\varphi_\mu = \frac{\partial \varphi}{\partial x_\mu}$. We can then write eq. 10 in general form by summing over the repeated indices ν and μ:

$$\Box\varphi = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\nu}\varphi = g^{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\varphi = g^{\mu\nu}\frac{\partial^2\varphi}{\partial x_\mu \partial x_\nu} = 0\,, \tag{11}$$

where the contravariant tensor $g^{\mu\nu}$ allows one to 'lower the indices' and use only covariant components. We can clearly see that "this expression reduces to the expression [eq. 10] since the derivatives with respect to different x^μ and x_ν are canceled by their coefficient" (Bachelard 1929, 69), which are $= 0$ for $\mu \neq \nu$. In doing so, "[w]e have only added ghost derivatives", which disappear by a suitable choice of the coordinates (69). However, thereby we have put eq. 10 in a form "that is both more general and more compact [*résumée*], more purely algebraic as well", since "geometric axes of the reference system", that is Galilean coordinates, "have lost all privilege" (69). The equation has the same form in all systems of coordinates.

The derivative of a scalar is a covariant vector $\varphi_\mu = \frac{\partial\varphi}{\partial x_\mu}$, the gradient. As we have seen, the derivative of this vector (i.e., the second derivative of φ), "will lose its tensorial nature. Indeed, we know that the derivative of a vector is not a tensor" (69). However, while using Galilean coordinates, we may replace the ordinary derivative with the covariant derivative of $\frac{\partial\varphi}{\partial x_\mu}$, which, as we have seen in eq. 8, is a covariant tensor of rank 2, $\varphi_{\mu\nu}$. Hence, eq. 10 takes the compact form:

$$g^{\mu\nu}\varphi_{\mu\nu} = 0\,, \tag{12}$$

where the mixed tensor $\varphi_{\mu\nu}$ is the covariant derivative of the covariant vector $\frac{\partial\varphi}{\partial x_\mu}$. This is a tensor equation: if it vanishes in a Galileian coordinate system, it vanishes in all coordinate systems. As Bachelard points out, "this process always refers to the same method. It specifically requires only the addition of Christoffel symbols that are identically zero in Galilean coordinates" (69). This is clearly evident if one goes a step beyond Bachelard and, following Becquerel (and Eddington), writes eq. 12 in a more fleshed-out form:

$$g^{\mu\nu}\varphi_{\mu\nu} \equiv g^{\mu\nu}\left(\overbrace{\underbrace{\frac{\partial^2\varphi}{\partial x_\mu \partial x_\nu}}_{\text{not a tensor}} - \underbrace{\begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}\frac{\partial\varphi}{\partial x_\alpha}}_{\text{not a tensor}}}^{\text{tensor}}\right) = 0\,. \tag{13}$$

If one uses Cartesian coordinates, the Christoffel symbols vanish and eq. 13 reduces to eq. 11. In non-Cartesian coordinates, the Christoffel symbols do not vanish and compensate for the change of coordinates. Thus, if the potential φ propagates according to the law eq. 10 in Galilean coordinates, it propagates according to eq. 13 in any coordinate systems: "This is indeed a purely formal complement which is not, in any way, included in the algebraic elements of the initial problem. No deduction, in particular, could necessarily lead us to it" (69).

Nevertheless, with eq. 12 "we have [...] acquired an entirely new property" (70). Since eq. 12 "now appears in tensor form" (70), $\varphi_{\mu\nu}$ is invariant. If $\varphi_{\mu\nu} = 0$ in Galilean coordinates, it must remain $= 0$ in any curvilinear coordinates in the case $R^{\rho}_{\mu\nu\sigma} = 0$: "So, from [eq. 10] to [eq. 12], one goes, algebraically speaking, from the particular to the general" (70), from an equation valid only in a special class of coordinate systems to an equation that holds in all possible coordinate systems. One might object that thereby only "a formal generality" has been achieved (70), since we have only put an equation in tensor form, while $R^{\rho}_{\mu\nu\sigma} = 0$. As we have mentioned, it is the equivalence principle that enables one to achieve "a generality of fact" (70). The equivalence principle guarantees that by putting a non-gravitational law in tensorial form for the case $R^{\rho}_{\mu\nu\sigma} = 0$, one obtains the corresponding gravitational law valid in the general case $R^{\rho}_{\mu\nu\sigma} \neq 0$: "This is precisely the case of the law expressed by [eq. 12]" (Bachelard 1929, 71) which only contains a term depending on the Christoffel symbols, but on the Riemann-Christoffel tensor. As Bachelard comments:

> We have thus found the differential equation, correct and complete, capable of determining the law of the propagation of the potential φ in the case where this propagation occurs through a gravitational field. In this way, algebra has been induced to cooperate with reality, with its own impulse towards calculation, without ever assuming and seeking instruction from reality as primary. To summarize, let's take an overall look at the stages of the construction. The problem was approached through its formal characteristics. Then the tensorial character, which was truly mutilated by the degeneration of certain variations, was sought. Once highlighted, this tensorial element, by itself, restored the law in its entirety. An invariant character then presented itself, allowing the transition from the particular case to the general case. Finally, the assertion of the principle of equivalence regulated the osculation of reality through the laboriously and progressively constructed general framework. (71)

Relativity theory provides is a sort of prescription for deriving gravitational laws from well-established special relativistic non-gravitational laws. Indeed, the heuristic strategy just described is not an "exceptional artifice"; on the contrary, it incorporates "a very characteristic rule of relativistic methods" (72).

One begins with the special-relativistic formulation of a law valid in a Galileian coordinate system, where the $g_{\mu\nu}$ have 'degenerate,' constant values and can then be omitted, and reformulates it as a 'tensor equation' in which generic $g_{\mu\nu}$ and their first derivatives appear explicitly. The equations now have the same form in all coordinate systems: "One could even posit at the center of relativistic thought a true principle of progress, which could be called *the principle of complete functionality*" (72). In other terms, the tensor calculus "re-establishes the diverse components" like some of the partial derivatives $\frac{\partial}{\partial x_\mu}$ and $\frac{\partial}{\partial x_\nu}$ in the wave equation eq. 11, that had been "simply erased due to the coordinate system used", that is, by choosing a Galileian coordinate system as in eq. 10. In this sense, tensor calculus "is truly a calculation that aims for generalization and generalizes by sensitizing all variables" (72); it allows one to put an equation that holds only in a special class of coordinate systems into a form that remains the same in all coordinate systems.

In this way, tensor calculus integrates "variations into reality, which, quantitatively speaking, might have initially seemed purely virtual" (73). Indeed, at first, one introduces only a 'formal generalization' since only values of the $g_{\mu\nu}$'s are considered that are obtainable from the Galilean values by means of coordinate transformations. One obtains a 'material generalization' by allowing for more general $g_{\mu\nu}$-systems, which are not reducible to the Galilean values by any choice of the coordinates. Einstein's theory of gravitation is based on the assumption that in both cases the non-Galilean values of the $g_{\mu\nu}$'s (or equivalently the non-vanishing of the Christoffel symbols) are connected with the phenomenon of gravitation. By this connection, the mere act of expressing equations in tensor form gains an 'inductive value': from special relativistic non-gravitational laws valid from $R^{\rho}_{\mu\nu\sigma} = 0$ one obtains gravitational laws valid in the general case $R^{\rho}_{\mu\nu\sigma} \neq 0$ by simply replacing ordinary with covariant derivatives. This inductive inference holds as long if one assumes that the gravitational laws contain only the first derivatives of the $g_{\mu\nu}$ (that is, they do not contain terms depending on the Riemann-Christoffel tensor $R^{\rho}_{\mu\nu\sigma}$).[18]

Following Becquerel (and Eddington), Bachelard treats the latter assumption as a reworded version of the equivalence principle (Becquerel 1922b, §§77-78). Bachelard rightly defines the latter as "one of the most surprising, most beautiful arguments of relativity" (Bachelard 1929, 74). Because of the identity of inertial and gravitational mass, the relativists identify a coordinate effect in the case $R^{\rho}_{\mu\nu\sigma} = 0$—the non-constancy of the components $g_{\mu\nu}$ due to the introduction of a uniformly accelerated coordinate system—with the presence of a homogeneous gravitational field (Norton 1985). Then they extend this identification to the general case in which the non-constancy of the $g_{\mu\nu}$'s cannot be eliminated by any coordinate transformation, $R^{\rho}_{\mu\nu\sigma} \neq 0$. Bachelard clearly understood that this extrapolation embodies "the inductive audacity" of relativity (Bachelard 1929, 74). Relativists boldly transfer what appeared to be only a redundancy in the *description* of the real to the real itself, at "the cost of placing the arbitrary on the same level as the real" (75). In doing so, they establish an equivalence between an 'artificial' (or 'geometrical,' in Becquerel's parlance) gravitational field with $R^{\rho}_{\mu\nu\sigma} = 0$ and a 'permanent' gravitational field $R^{\rho}_{\mu\nu\sigma} \neq 0$. They can then extend this equivalence to all laws of nature that involve only the $g_{\mu\nu}$'s and their first derivatives (see also 130–134).

Conclusion

For Bachelard, the 'inductive value' of general relativity is ultimately based on the following principle: All laws of nature governing phenomena in an artificial gravitational field, which depend on the $g_{\mu\nu}$'s and their first derivatives, will also hold in a permanent gravitational field (Eddington 1920a, 43). Relativity does not provide a *deductive* proof of this assertion. Indeed, laws which depend on the second derivatives of the $g_{\mu\nu}$'s are not logically incompatible with relativity. Relativity uses this requirement as an *inductive* recipe to obtain gravitational laws.[19] According to Bachelard, in relativity such "an induction is promoted to the rank of a method" (Bachelard 1929, 75): (1) start with the equation expressing a law of nature in which only ordinary derivatives appear; (2) put it in its tensor form by substituting ordinary derivatives with covariant derivatives. In this way, one discovers "in a particular case, not only the features of generality but also the paths to generalization, moving from an immanent generality to a transcendent generality" (75), transitioning from special relativistic non-gravitational laws to general-relativistic gravitational laws. As one can see, this mathematical

[18]For a recent discussion, see Read, Brown, and Lehmkuhl (2018).

[19]The principle of 'minimal coupling' in modern terms.

generalization is nothing but the modern comma-goes-to-semicolon rule.[20]

Bachelard's characterization of this method as a form of 'relativistic induction' was clearly meant to highlight the central message of the book by contrasting it with Meyerson's 'relativistic deduction,' with which most of his potential readers were probably familiar. However, Bachelard's 'promotional strategy' fell flat. Besides the 1928 glowing review of the book (Einstein 1928), in early February 1929, Einstein mentioned Meyerson approvingly in a popular article published in both the *New York Times* (Einstein 1929a) and the *London Times* (Einstein 1929b). One can question whether Einstein's endorsement was based on a genuine understanding of Meyerson's philosophy (Giovanelli 2018). However, it was hard to avoid the impression that Einstein himself authoritatively emphasized the 'deductive value' of relativity rather than its 'inductive value.' .

Bachelard's book received only a few reviews (Rabeau 1929; A. Bc. 1929; Anonymous 1930; Metzger 1930; Spaier 1931–32), without stirring a particularly significant discussion. When he became a professor at the University of Dijon in 1930, both the physics and epistemological debates in France were soon captured by the new quantum theory (Langevin 1931; Meyerson 1933). Bachelard's (1932) interests also moved from macro- to microphysics as he developed the ideas of 'phenomenotechnique' and 'noumenology,' which subsequently became central to his philosophy (Chimisso 2008; Fabry 2019). Yet, he appeared to have kept the central tenet of his interpretation of relativity unchanged. The second chapter of his next major monograph, *Le nouvel esprit scientifique,* opens with a reference to his 1929 booklet and presents relativity as an example of the 'new scientific mind' (Bachelard 1934, 41). Once again, Bachelard insists on the role of tensor calculus as a genuine method of invention: 'tensor calculus knows physics better than the physicist does,' as Langevin once put it (54). The rules of index manipulation allow one to check whether an equation is written in coordinate-free form and, at the same time, reveal new physical insights, the gravitational effects on phenomena.

The reasons for 'Bachelard's silence' (Fruteau de Laclos 2005) on relativity in subsequent years have been a matter of debate among scholars (Parrocchia 2014; Alunni 2019). However, the outcome of this debate is inconsequential for understanding Bachelard's contribution to the history of philosophical interpretations of relativity. When Bachelard returned to writing on relativity over decade later in his essay (Bachelard 1949) for Einstein's volume in the Library of Living Philosophers, his stance appears to remain unchanged. Langevin's motto, that tensor calculus knows relativity better than the relativist himself, once again served as a witty summary of his interpretation of relativity (578). In the closing of the volume, Einstein (1949) did not comment on Bachelard's essay. Bachelard's philosophical jargon was probably alien to him.[21] Nevertheless, Bachelard seems to have grasped a fundamental point that was characteristic of Einstein's own reading of the theory. As Einstein emphasized in his later years, the true conceptual core of general relativity lies in the possibility of "tensor formation by differentiation in relation to an arbitrary coordinate system" (Einstein to Besso, Aug. 10, 1954; Speziali 1972, Doc. 210). Bachelard was the only

[20]E.g., the four-dimensional inhomogeneous Maxwell's equations $F^{\mu\nu}_{,\nu} = -\frac{4\pi}{c} j^{\mu}$ become $F^{\mu\nu}_{;\nu} = -\frac{4\pi}{c} J^{\mu}$. The ; implies the Christoffel symbols, incorporating the first derivatives of the $g_{\mu\nu}$'s, showing how the gravitational field affects the electromagnetic.

[21]See, Einstein's unpublished comment (EA, 2060) about Andrew Paul Ushenko's essay (Schilpp 1949, 609–645).

participant in the early philosophical debate on relativity who thoroughly grasped this crucial point.

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